BLOWING-UPS DESCRIBING THE POLARIZATION CHANGE OF MODULI SCHEMES OF SEMISTABLE SHEAVES OF GENERAL RANK

KIMIKO YAMADA

Abstract. Let H and H' be two ample line bundles over a smooth projective surface X, and $M(H)$ (resp. $M(H')$) the coarse moduli scheme of H-semistable (resp. H'-semistable) sheaves of fixed type (r, c_1, c_2). We construct a sequence of blowing-ups which describes how $M(H)$ differs from $M(H')$ not only when $r = 2$ but also when r is arbitrary. Means we here utilize are elementary transforms and the notion of a sheaf with flag.

1. Introduction

Let X be a nonsingular projective surface over an algebraically closed field k with character zero, H_- and H_+ two ample line bundles over X, and $\mathfrak{c} = (r, c_1, c_2)$ an element of $\mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}$. There exists the coarse moduli scheme $M(H_-, \mathfrak{c})$, which is projective over k, of S-equivalence classes of H_--semistable sheaves E on X such that $(r(E), c_1(E), c_2(E)) = \mathfrak{c}$ by [4]. When $2rc_2 - (r-1)c_1^2$ is sufficiently large, $M(H_-, \mathfrak{c})$ and $M(H_+, \mathfrak{c})$ are birationally equivalent from [7, Theorem 4.C.7]. With this in mind, we shall construct a sequence of morphisms

$$\begin{tikzcd}
 & \tilde{M}_0 \arrow[dl, "p_0"'] \arrow[dr, "q_0"] & & \cdots \arrow[dl, "p_1"'] & \tilde{M}_{m-1} \arrow[dr, "q_{m-1}"] & \\
M(\Delta_0, \mathfrak{c}) \arrow[d, "h"] & & M(\Delta_1, \mathfrak{c}) & & & M(\Delta_m, \mathfrak{c}) \arrow[d, "h"] \\
M(H_-, \mathfrak{c}) & & & & & M(H_+, \mathfrak{c})
\end{tikzcd} \tag{1}$$

assuming that H_- and H_+ lie in adjacent chambers ([21, Definition 2.1]) of type $\mathfrak{c}$.

To execute our purpose we utilize elementary transforms and introduce a sheaf with flag, or a SF for short. Elementary transforms have appeared in the study of the polarization change problem for stability conditions of rank-two stable sheaves. However we can not directly apply this way to the general-rank case partly because an H_--semistable and not H_+-semistable sheaf of type $\mathfrak{c}$ is H_--stable if its rank is two, but it is not necessarily H_--stable in general. For example, if a sheaf F of type $\mathfrak{c}$ is H_--semistable and not H_+-semistable, then $F \oplus F$ is H_--semistable, not H_+-semistable and not H_--stable. It is unfavorable since the complement of $M^s(H_-, \mathfrak{c}) \subset M(H_-, \mathfrak{c})$, the open set of all H_--stable sheaves, is complicated. Hence in Section 2 we introduce a sheaf with flag (SF) and its Δ-stability with respect to

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(L, C), where Δ is a parameter, L a line bundle on X and $C \subset X$ an effective divisor. As is discussed in Section 3 the coarse moduli scheme of Δ-semistable SFs exists; $M(\Delta, \mathfrak{c})$ at (1) is deduced from it. Corollary 2.7 shows that under some condition the problem of observing how stability conditions of SFs vary as parameters Δ do is similar to the polarization change problem for rank-two stable sheaves. With this corollary as a base, we obtain blowing-ups p_i and q_i at (1), whose centers are topics in Section 4. The morphism $h : M(\Delta_0, \mathfrak{c}) \to M(H_-, \mathfrak{c})$ at (1) is naturally induced when Δ_0 and (L, C) are chosen appropriately. Its restriction $h : h^{-1}(M^s(H_-, \mathfrak{c})) \to M^s(H_-, \mathfrak{c})$ is a Grassmannian-bundle in étale topology.

Here let us mention the background. With its relation to the wall-crossing formula of Donaldson polynomials, the polarization change problem for stability conditions of sheaves has been a subject of study. Matsuki-Wentworth [12] pointed out in general-rank case that this problem is a subject concerning how the GIT quotient of a quasi-projective scheme S by a reductive group G varies as G-linearized ample line bundles of S do, and connected $M(H_-)$ and $M(H_+)$ by a sequence of Thaddeus-type flips ([18]). On the other hand, elementary transforms, refer to [10] and [2, Appendix] about general information, has the following advantages: (i) birational transforms obtained there are blowing-ups whose centers are derived by a canonical, moduli-theoretic way; (ii) when two parameters α and α' defining stability conditions of objects are given, one not only connects the moduli scheme of α-semistable objects with that of α'-semistable ones, but also relates their universal families, if exist. Ellingsrud-Göttsche [1] and Friedman-Qin [3] proposed to apply elementary transform to the case where $r = 2$ and the wall of type $\mathfrak{c}$ separating H_- and H_+ is good, so the natural subset

$$M(H_-, \mathfrak{c}) \supset P = \big\{[E] \bigm| E \text{ is not } H_+\text{-semistable}\big\}$$

is relatively easy to handle. These papers have stimulated the author to write this article. The author aims to consider this problem with no restriction on this wall. As a result this subset P unkindly behaves in general, and we have to observe its (infinitesimal) structure in more detail. The preceding paper [20] dealt with rank-two case and this article the case where r is arbitrary, and hence we need further devices explained above. We finally note that while writing this article the author found that also Mochizuki used the notion of sheaves with flag in [14], where he considered not birational transforms describing the variation of moduli schemes, but the wall-crossing formula of Donaldson polynomials in general-rank case.

The content of this article is as follows. In Section 2 we define some basic terms and show Corollary 2.7 mentioned above. In Section 3 we construct the moduli scheme $M(\Delta, \mathfrak{f})$ of Δ-semistable SFs of type $\mathfrak{f}$ and study its infinitesimal structure. The scheme $M(\Delta, \mathfrak{c})$ at (1) is the union of some connected components of $M(\Delta, \mathfrak{f})$. We focus in Section 4 on the subscheme $P \subset M(\Delta_-, \mathfrak{f})$ consisting of SFs which are Δ_--semistable and not Δ_+-semistable, and discuss its relative obstruction theory. In Section 5 we arrive at the sequence (1) by elementary transforms.

Notation. X is a smooth projective surface over an algebraically closed field k with character zero. For k-schemes S and T, T_S means $T \times S$ and $p_T : T_S \to S$ is the natural projection. $\mathcal{H}^i(A)$ is the i-th cohomology of $A \in D(T) := D(Qcoh(T))$. $Mc(A \to B)$ is the mapping cone of a morphism $A \to B$ in $D(T)$. $\underline{\otimes}$ stands for the derived functor of $\otimes$.

2. Sheaves with flag

Definition 2.1. *A sheaf with flag (SF) of length* n is a pair $\mathcal{E} = (E, \{\Gamma_i\}_{i=1}^n)$ consisting of a coherent sheaf E on X and a flag of vector spaces $\Gamma_1 \subset \Gamma_2 \subset \dots \Gamma_n \subset H^0(E)$. *A homomorphism* $f : \mathcal{E}' = (E', \Gamma'_\bullet) \to \mathcal{E} = (E, \Gamma_\bullet)$ *of SFs of length* n is a homomorphism $f : E' \to E$ of sheaves which preserves their flag structures. $\mathrm{Hom}_{SF}(\mathcal{E}', \mathcal{E})$ denotes the set of all homomorphisms $f : \mathcal{E}' \to \mathcal{E}$ of SFs. We say that a sequence $\mathcal{E}^{(0)} \xrightarrow{f^{(0)}} \mathcal{E}^{(1)} \xrightarrow{f^{(1)}} \mathcal{E}^{(2)}$ of SFs $\mathcal{E}^{(j)} = (E^{(j)}, \Gamma_\bullet^{(j)})$ and homomorphisms is *exact* if both $E^{(0)} \to E^{(1)} \to E^{(2)}$ and $\Gamma_i^{(0)} \to \Gamma_i^{(1)} \to \Gamma_i^{(2)}$ (i is arbitrary) are exact. *A sub SF* $\mathcal{E}' \subset \mathcal{E}$ is given by a homomorphism $\iota : \mathcal{E}' \to \mathcal{E}$ of SFs such that $\iota : E' \to E$ is injective. A sub SF $\mathcal{E}' = (E', \Gamma'_\bullet) \subset \mathcal{E} = (E, \Gamma_\bullet)$ is said to be *saturated* if the induced homomorphism $\Gamma_i / \Gamma'_i \to H^0(E/E')$ is injective for all i; in other words, $\Gamma'_i = H^0(E') \cap \Gamma_i$ for all i. In this case, also $\mathcal{E}/\mathcal{E}' = (E/E', \{\Gamma_i/\Gamma'_i\}_i)$ is a SF. A SF $\mathcal{E} = (E, \Gamma_\bullet)$ of length n is *full* if it holds that $\mathrm{rk}\Gamma_i = i$ for all i and that $n = h^0(E)$.

Definition 2.2. Let $\mathcal{O}(1)$ be an ample line bundle on X, L a line bundle on X, and $C \subset X$ an effective divisor on X. A sheaf E is said to be *of type* $\mathfrak{f}' \in \mathbb{Q}[l]^{\times 3}$ if

$$\big(\chi(E(l)),\, \chi(E \otimes L(-C)(l)),\, \chi(E \otimes L(l))\big) = \mathfrak{f}',$$

and a SF $\mathcal{E}$ of length n is said to be *of type* $\mathfrak{f} \in \mathbb{Q}[l]^{\times 3} \times \mathbb{Z}^{\times n}$ if

$$\big(\chi(E(l)),\, \chi(E \otimes L(-C)(l)),\, \chi(E \otimes L(l)),\, \mathrm{rk}\Gamma_1, \dots, \mathrm{rk}\Gamma_n\big) = \mathfrak{f}.$$

When a parameter $\Delta = (a, \delta_1, \dots, \delta_n) \in (0,1) \times \mathbb{Q}_{>0}^{\times n}$ is given, we also define the *reduced Hilbert polynomial* of a SF $\mathcal{E}$ of length n with $\mathrm{rk}E > 0$ by

$$p^\Delta(\mathcal{E})(l) = \frac{1}{\mathrm{rk}E}\left\{(1-a)\chi\left(E \otimes L(-C)(l)\right) + a\chi\left(E \otimes L(l)\right) + \sum_{i=1}^n \delta_i \cdot \mathrm{rk}\Gamma_i\right\} \in \mathbb{Q}[l].$$

Definition 2.3. For a parameter $\Delta = (a, \delta_1, \dots, \delta_n) \in (0,1) \times \mathbb{Q}_{>0}^{\times n}$, we say that a SF $\mathcal{E} = (E, \Gamma_\bullet)$ of length n is Δ*-stable* (*resp. semistable*) if E is torsion-free and it holds that $p^\Delta(\mathcal{E}') < p^\Delta(\mathcal{E})$ (resp. $\leq$) for any proper sub SF $\mathcal{E}' \subset \mathcal{E}$. We define the *S-equivalence* of Δ-semistable SFs in the same way as the case of semistable sheaves [7, p. 22].

For $\mathfrak{f} \in Q[l]^{\times 3} \times \mathbb{Z}^{\times n}$, we set $\mathcal{S}_1(\mathfrak{f})$ to be the set of all nonzero SFs $\mathcal{E}'$ of length n such that there are a SF $\mathcal{E} = (E, \Gamma_\bullet)$ of type $\mathfrak{f}$ and a parameter Δ_0 satisfying (i) E is $\mathcal{O}(1)$-semistable, (ii) $\mathcal{E}'$ is a proper sub SF of $\mathcal{E}$ and (iii) $p^{\Delta_0}(\mathcal{E}') = p^{\Delta_0}(\mathcal{E})$. Any SF $\mathcal{E}' \in \mathcal{S}_1(\mathfrak{f})$ gives a subset in $(0,1) \times \mathbb{Q}_{>0}^{\times n}$

$$W(\mathcal{E}', \mathfrak{f}) = \left\{\Delta = (a, \delta_\bullet) \;\middle|\; p^\Delta(\mathcal{E}') = p^\Delta(\mathcal{E}) \text{ for any SF } \mathcal{E} \text{ of type } \mathfrak{f} \right\}.$$

Grothendieck's lemma on boundedness [7, p. 29] implies $\left\{W(\mathcal{E}', \mathfrak{f}) \;\middle|\; \mathcal{E}' \in \mathcal{S}_1(\mathfrak{f})\right\}$ is finite.

Definition 2.4. This $W(\mathcal{E}', \mathfrak{f})$ is called a *SF-wall of type* $\mathfrak{f}$ if it is a proper subset of $(0,1) \times \mathbb{Q}_{>0}^{\times n}$. *A SF-chamber of type* $\mathfrak{f}$ is a connected component of the complement of the union of all SF-walls of type $\mathfrak{f}$. Δ-semistability of SFs of type $\mathfrak{f}$ does not change unless Δ passes through a SF-wall of type $\mathfrak{f}$.

We say that a $\mathcal{O}_X$-module F has *the property* (O) (resp. (O_m)) with respect to an ample line bundle $\mathcal{O}(1)$ if F (resp. $F(m)$) is generated by global sections and

its higher cohomologies vanish. For $\mathfrak{f}' \in \mathbb{Q}[l]^{\times 3}$ we define two families as follows: let $\mathcal{S}_2(\mathfrak{f}')$ be the set of all $\mathcal{O}(1)$-slope-semistable sheaves of type $\mathfrak{f}'$ on X, and let $\mathcal{S}_3(\mathfrak{f}')$ be the set of all sheaves E' on X such that E' is a subsheaf of a certain $E \in \mathcal{S}_2(\mathfrak{f}')$ with torsion-free quotient and satisfies $\mu_{\mathcal{O}(1)}(E') = \mu_{\mathcal{O}(1)}(E)$. If one replace E with $E(m)$ where m is sufficiently large, then it holds that

$$\text{Every member of } \mathcal{S}_2(\mathfrak{f}') \cup \mathcal{S}_3(\mathfrak{f}') \text{ has the property } (O). \tag{2}$$

Definition 2.5. We say that $\mathfrak{f} = (\mathfrak{f}', l_1, \dots, l_n) \in \mathbb{Q}[l]^{\times 3} \times \mathbb{Z}^{\times n}$ has *the property* (A) if (2) is valid for $\mathfrak{f}' = (f, f_0, f_1)$ and if n and $\mathfrak{f}$, respectively, equal $f(0)$ and $(\mathfrak{f}', 1, 2, \dots, n)$.

One can verify that if $\mathfrak{f}$ has the property (A) and if a parameter Δ is contained in no SF-wall of type $\mathfrak{f}$, then a Δ-semistable SF $\mathcal{E}$ of type $\mathfrak{f}$ is always Δ-stable. Moreover, we have the proposition below.

Proposition 2.6. *Assume that $\mathfrak{f}$ has the property (A). Suppose that a parameter Δ_0 is contained in just one SF-wall of type $\mathfrak{f}$ and a SF $\mathcal{E}$ of type $\mathfrak{f}$ is Δ_0-semistable. If a proper sub SF $\mathcal{E}' = (E', \Gamma'_\bullet) \subset \mathcal{E} = (E, \Gamma_\bullet)$ satisfies $p^{\Delta_0}(\mathcal{E}') = p^{\Delta_0}(\mathcal{E})$, then $\mathcal{E}'$ is saturated and both $\mathcal{E}'$ and $\mathcal{E}/\mathcal{E}'$ are Δ_0-stable.*

Proof. Remark that if a SF $\mathcal{E}$ of type $\mathfrak{f}$ is semistable with respect to some parameter Δ then $\mathcal{E}$ becomes full. $\mathcal{E}'$ clearly is saturated and Δ_0-semistable. If $\mathcal{E}'$ is not Δ_0-stable, then there is a proper sub SF $\mathcal{E}'' = (E'', \Gamma''_\bullet)$ of $\mathcal{E}'$ such that $p^{\Delta_0}(\mathcal{E}) = p^{\Delta_0}(\mathcal{E}') = p^{\Delta_0}(\mathcal{E}'')$. This implies both $W(\mathcal{E}', \mathfrak{f})$ and $W(\mathcal{E}'', \mathfrak{f})$ are SF-wall containing Δ_0, so $W(\mathcal{E}', \mathfrak{f})$ equals $W(\mathcal{E}'', \mathfrak{f})$. Thus we find a constant λ such that

$$p^{\Delta}(\mathcal{E}) - p^{\Delta}(\mathcal{E}') = \lambda \left\{ p^{\Delta}(\mathcal{E}) - p^{\Delta}(\mathcal{E}'') \right\}$$

for all Δ. One can deduce that

$$\frac{i}{\mathrm{rk}E} - \frac{\mathrm{rk}\Gamma'_i}{\mathrm{rk}E'} = \lambda \left\{ \frac{i}{\mathrm{rk}E} - \frac{\mathrm{rk}\Gamma''_i}{\mathrm{rk}E''} \right\}$$

for all i, which means that

$$\frac{1}{\mathrm{rk}E} - \frac{\mathrm{rk}(\Gamma'_i/\Gamma'_{i-1})}{\mathrm{rk}E} = \lambda \left\{ \frac{1}{\mathrm{rk}E} - \frac{\mathrm{rk}(\Gamma''_i/\Gamma''_{i-1})}{\mathrm{rk}E''} \right\} \tag{3}$$

for all i. Since $\mathcal{E}$ is full, $\mathrm{rk}(\Gamma'_i/\Gamma'_{i-1})$ is either 0 or 1. If $\mathrm{rk}(\Gamma'_i/\Gamma'_{i-1})$ equals 1 for all i then it follows that $H^0(E') = H^0(E') \cap \Gamma_n = \Gamma'_n = \Gamma_n = H^0(E)$. This is contradiction since E is generated by global sections from (2). Accordingly

$$\mathrm{rk}(\Gamma'_{i_0}/\Gamma'_{i_0-1}) = 0 \quad \text{for some } i_0. \tag{4}$$

As to this i_0, one can check that

$$\mathrm{rk}(\Gamma''_{i_0}/\Gamma''_{i_0-1}) = 1. \tag{5}$$

From (3), (4) and (5) we can determine λ and hence show that

$$\mathrm{rk}(\Gamma'_i/\Gamma'_{i-1}) + \mathrm{rk}(\Gamma''_i/\Gamma''_{i-1}) = 1 \quad \text{for all } i. \tag{6}$$

On the other hand, $H^0(E'') \neq 0$ by (2), so there is a nonzero section $\tau \in H^0(E'')$. Since $\mathcal{E}$ is full, some j enjoys the property that

$$\tau \notin H^0(E'') \cap \Gamma_{j-1} = \Gamma''_{j-1} \quad \text{and that} \quad \tau \in H^0(E'') \cap \Gamma_j = \Gamma''_j.$$

As to this j, it also holds that

$$\tau \notin H^0(E') \cap \Gamma_{j-1} = \Gamma'_{j-1} \quad \text{and that} \quad \tau \in H^0(E') \cap \Gamma_j = \Gamma'_j.$$

However these facts contradict (6). Therefore $\mathcal{E}'$ is Δ_0-stable. $\square$

Corollary 2.7. *Assume that $\mathfrak{f}$ has the property (A), two parameters Δ_- and Δ_+ are contained in adjacent SF-chambers of type $\mathfrak{f}$, and that $\Delta_0 = t\Delta_- + (1-t)\Delta_+$ $(0 < t < 1)$ is contained in just one SF-wall of type $\mathfrak{f}$. Suppose that a SF $\mathcal{E}$ of type $\mathfrak{f}$ is Δ_--semistable and not Δ_+-semistable and hence there is an exact sequence of SFs*

$$0 \longrightarrow \mathcal{F}^{(l)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}^{(r)} \longrightarrow 0,$$

where $\mathcal{F}^{(l)}$ is saturated and satisfies $p^{\Delta_+}(\mathcal{F}^{(l)}) > p^{\Delta_+}(\mathcal{E})$. (We call such a sub SF $\mathcal{F}^{(l)}$ a Δ_+-destabilizer *of $\mathcal{E}$.) Then the following holds:*

(i) *$\mathcal{E}$ is Δ_--stable, and its Δ_+-destabilizer is unique.*

(ii) *If a SF $\mathcal{E}'$ is endowed with a nontrivial exact sequence*

$$0 \longrightarrow \mathcal{F}^{(r)} \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}^{(l)} \longrightarrow 0,$$

then $\mathcal{E}'$ is Δ_+-semistable.

Proof. The definition of SF-chambers deduces that $p^{\Delta_0}(\mathcal{F}^{(l)}) = p^{\Delta_0}(\mathcal{E})$ and that any Δ_+-destabilizer $\mathcal{F}$ of $\mathcal{E}'$, if it exists, satisfies that $p^{\Delta_0}(\mathcal{F}) = p^{\Delta_0}(\mathcal{E}')$. This corollary follows these facts and the lemma above. $\square$

3. Moduli theory of SFs

Let us begin with the construction of the coarse moduli scheme of semistable SFs. We fix $\mathfrak{f} = (\mathfrak{f}' = (f, f_0, f_1), r_1, \dots, r_n)$.

Definition 3.1. For a scheme S over k, a *S-flat family of SFs* on X is a pair $(E_S, \{\Gamma_{i,S}\}_i)$ consisting of a S-flat sheaf E_S on X_S and a sequence of quotients

$$Ext^2_{X_S/S}(E_S, K_X) \twoheadrightarrow (\Gamma_{n,S})^\vee \twoheadrightarrow \cdots \twoheadrightarrow (\Gamma_{1,S})^\vee,$$

where $\Gamma_{i,S}$ is a locally-free $\mathcal{O}_S$-module. *A homomorphism $f : \mathcal{E}'_S = (E'_S, \Gamma'_{\bullet,S}) \to \mathcal{E}_S = (E_S, \Gamma_{\bullet,S})$ of flat families of SFs* of length n is a homomorphism $f : E'_S \to E_S$ which induces a homomorphism $f_i : \Gamma'_{i,S} \to \Gamma_{i,S}$ $(1 \le i \le n)$ such that

$$\begin{tikzcd} Ext^2_{X_S/S}(E_S, K_X) \arrow[r, "f"] \arrow[d] & Ext^2_{X_S/S}(E'_S, K_X) \arrow[d] \\ (\Gamma_{i,S})^\vee \arrow[r, "f_i^\vee"] & (\Gamma'_{i,S})^\vee \end{tikzcd}$$

is commutative.

Define a functor $\underline{M} : (\mathrm{Sch}/k)^\circ \to (\mathrm{Sets})$ as follows. We first set $\underline{M}'(S)$ is to be the set of all S-flat families of Δ-semistable SFs of type $\mathfrak{f}$ on X and $\underline{M}(S)$ is the quotient $\underline{M}'(S)/\sim$, where S-flat families $\mathcal{E}_S$ and $\mathcal{F}_S$ are equivalent if and only if it holds that $\mathcal{E}_S \otimes L \simeq \mathcal{F}_S$ for some line bundle L on S. We also define a functor $\underline{M}^s$ by replacing "Δ-semistable" with "Δ-stable" here.

Proposition 3.2. *The functor $\underline{M}$ has the coarse moduli scheme $M(\Delta, \mathfrak{f})$ which is projective over k. $M(\Delta, \mathfrak{f})(k)$ coincides with the set of all S-equivalence classes of Δ-semistable SFs of type $\mathfrak{f}$. Some open subset $M^s(\Delta, \mathfrak{f}) \subset M(\Delta, \mathfrak{f})$ is the coarse moduli scheme of the functor $\underline{M}^s$.*

Proof. One can prove this proposition in a similar fashion to Simpson's construction of moduli schemes of semistable sheaves [17] and [7, Chap. 4]. We also take the case of parabolic sheaves [11] and of coherent systems [6] as models.

First, there is an integer m such that if F belongs to $\mathcal{S}_2(\mathfrak{f}') \cup \mathcal{S}_3(\mathfrak{f}')$, then both F, $F \otimes L$, $F \otimes L(-C)$ and L have the property (O_m). Let V_m be a $f_1(m)$-dimensional vector space, and denote by $Q(m, \mathfrak{f}')$ Grothendieck's Quot-scheme parametrizing quotient $\mathcal{O}_X$-modules of $V_m \otimes L^{-1}(-m)$ whose type is $\mathfrak{f}'$, and by $U \subset Q(m, \mathfrak{f}')$ the open subset of all quotients $q : V_m \otimes L^{-1}(-m) \twoheadrightarrow E$ such that E is torsion-free, both E, $E \otimes L$ and $E \otimes L(-C)$ have the property (O_m), and $H^0(q) : V_m \to H^0(E \otimes L(m))$ is injective. $Q(m, \mathfrak{f}')$ has a universal family $V_m \otimes \mathcal{O}_{X_Q} \twoheadrightarrow E_Q \otimes L(m)$ on $X_{Q(m,\mathfrak{f}')}$.

Next, consider a functor $\underline{Fl}\left(Ext^2_{X_U/U}(E_U, K_X), r_\bullet\right) : (\mathrm{Sch}/U)^\circ \to (\mathrm{Sets})$ which associates with $S \to U$ the set of all sequences of surjective homomorphisms

$$Ext^2_{X_U/U}(E_U, K_X) \underset{U}{\otimes} \mathcal{O}_S \twoheadrightarrow (\Gamma_{n,S})^\vee \twoheadrightarrow \cdots \twoheadrightarrow (\Gamma_{1,S})^\vee$$

consisting of locally-free $\mathcal{O}_S$-modules $\Gamma_{i,S}$ with rank r_i. This is represented by a U-scheme, say R_m. By the choice of U a natural map $Ext^2_{X_U/U}(E_U \otimes L(m), K_X) \otimes H^0(L(m)) \to Ext^2_{X_U/U}(E_U, K_X)$ is surjective and $Ext^2_{X_U/U}(E_U \otimes L(m), K_X) \to Ext^2_{X_U/U}(V_m \otimes \mathcal{O}_{X_U}, K_X) \simeq V_m^\vee \otimes \mathcal{O}_U$ is isomorphic. Thus if we put $B_m = H^0(L(m))$ then R_m is embedded in $U \times Fl(V_m^\vee \otimes B_m, r_\bullet)$, where $Fl(V_m^\vee \otimes B_m, r_\bullet)$ is the flag scheme parametrizing sequences of surjective maps $V_m^\vee \otimes B_m \twoheadrightarrow \Gamma_n^\vee \twoheadrightarrow \cdots \twoheadrightarrow \Gamma_1^\vee$ consisting of vector spaces Γ_i with rank r_i.

Last, a natural map $Ext^2_{X_U/U}(E_U \otimes L(m)|_C, K_X) \to Ext^2_{X_U/U}(E_U \otimes L(m), K_X) \simeq V_m^\vee \otimes \mathcal{O}_U$ induces a morphism $U \to Gr\left(V_m, f_1(m) - f_0(m)\right) =: Gr(f_1 - f_0)$ to the Grassmannian parametrizing quotient vector spaces of V_m with rank $f_1(m) - f_0(m)$. Hence we obtain a embedding $R_m \subset U \times Fl(V_m^\vee \otimes B_m, r_\bullet) \times Gr(f_1 - f_0)$ and its closure

$$\overline{R_m} \subset Q(m, \mathfrak{f}') \times Fl(V_m^\vee \otimes B_m, r_\bullet) \times Gr(f_1 - f_0) \tag{7}$$

which is invariant under the natural action of $G = \mathrm{SL}(V_m)$.

Remember some G-linearized line bundles on the right side of (7);

$$\mathcal{O}^l_{Q,0}(1) = \det Rp_{X*}(E_Q \otimes L(-C)(l)) \text{ and } \mathcal{O}^l_{Q,1}(1) = \det Rp_{X*}(E_Q \otimes L(l))$$

are G-linearized ample line bundles on $Q(m, \mathfrak{f}')$ when l is sufficiently large [7, Prop. 2.2.5]. $Fl(V_m^\vee \otimes B_n, r_\bullet)$ has a universal family

$$V_m^\vee \otimes B_m \otimes \mathcal{O}_{Fl} \twoheadrightarrow (\Gamma_{n,Fl})^\vee \twoheadrightarrow \cdots \twoheadrightarrow (\Gamma_{1,Fl})^\vee . \tag{8}$$

For positive integers $k_1, \dots, k_n$,

$$\mathcal{O}_{Fl}(k_1, \dots, k_n) = \otimes_{i=1}^n \left(\det \Gamma^\vee_{i,Fl}\right)^{\otimes k_i}$$

is a G-linearized ample line bundle on $Fl(V_m^\vee \otimes B_m, r_\bullet)$ by the Plücker embedding. Similarly, if we denote a universal family of $Gr(f_1 - f_0)$ by $V_m \otimes \mathcal{O}_{Gr} \twoheadrightarrow W_{Gr}$,

then $\mathcal{O}_{Gr}(1) = \det W_{Gr}$ is a G-linearized ample line bundle on $Gr(f_1 - f_0)$. For a parameter $\Delta = (a, \delta_\bullet)$ we put $f^\Delta(l) = (1-a)f_0(l) + af_1(l) + \sum_{i=1}^n \delta_i \cdot r_i$ and then

$$L_l = \mathcal{O}^l_{Q,0}\left((1-a)f^\Delta(m)\right) \otimes \mathcal{O}^l_{Q,1}\left(af^\Delta(m)\right) \otimes \\ \mathcal{O}_{Fl}\left(\delta_1(f^\Delta(l) - f^\Delta(m)), \dots, \delta_n(f^\Delta(l) - f^\Delta(m))\right) \otimes \mathcal{O}_{Gr}((1-a)f^\Delta(l))$$

is a G-linearized $\mathbb{Q}$-ample line bundle on R_m when l is sufficiently large. The GIT quotient $\overline{R_m}^{ss}(L_l)//G$ ($\overline{R_m}^{s}(L_l)/G$, resp.) is the moduli scheme $M(\Delta, \mathfrak{f})$ ($M^s(\Delta, \mathfrak{f})$, resp.) if m is sufficiently large and if l is sufficiently large with respect to m. Its proof proceeds in a similar fashion to that of Theorem 4.3.3 in [7], so is left to the reader. □

Proposition 3.3. *If $\mathfrak{f}$ has the property (A), then $M^s(\Delta, \mathfrak{f})$ represents the functor $\underline{M}^s$.*

Proof. The sheaves E_U and $\Gamma_{i,Fl}$ in the proof of Proposition 3.2 give a flat family of SFs $\mathcal{E}_{R^s}$ over $R^s := R^s_m(L_l)$. On the other hand one can check that $R^s \to M^s = M^s(\Delta, \mathfrak{f})$ is a $PGL(V_m)$-bundle in a similar fashion to Proposition 6.4 in [9]. Because $\lambda \cdot \mathrm{id} \in GL(V_m)$ acts on the line bundle $\Gamma_{1,Fl}$ at (8) by the multiplication of λ,

$$\mathcal{E}_{R^s} \otimes \Gamma^\vee_{1,Fl} = \big(E_U \otimes \Gamma^\vee_{1,Fl}\,, \\ Ext^2_{X_{R^s}/R^s}(E_U \otimes \Gamma^\vee_{1,Fl}, K_X) \to \Gamma^\vee_{n,Fl} \otimes \Gamma_{1,Fl} \to \cdots \to \Gamma^\vee_{1,Fl} \otimes \Gamma_{1,Fl}\big) \tag{9}$$

descends to a $M^s(\Delta, \mathfrak{f})$-flat family $\overline{\mathcal{E}}_{M^s} = (\overline{E}_{M^s}, \overline{\Gamma}_{\bullet,M^s})$ from fpqc descent theory. By the assumption $p_{X*}(\overline{E}_{M^s}) =: \overline{V}_{M^s}$ is a vector bundle on M^s endowed with a flag structure. After [7, p. 49] we denote by $Hom^-_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s}) \subset Hom_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s})$ the subsheaf consisting of all homomorphisms which preserves the flag structure, and by $Hom^+_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s})$ its quotient. By $\mathrm{id} \in \mathrm{End}(E_{M^s})$ and a natural map

$$RHom_{X_{M^s}/M^s}(\overline{E}_{M^s}, \overline{E}_{M^s}) \longrightarrow Hom_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s}) \longrightarrow Hom^+_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s}) \tag{10}$$

we obtain two triangles

$$\mathcal{O}_{M^s} \longrightarrow K^0_{M^s}[-1] \longrightarrow Mc(\mathrm{id}) \xrightarrow{+1} \quad \text{and} \tag{11}$$

$$RHom_{X_{M^s}/M^s}(\overline{E}_{M^s}, \overline{E}_{M^s}) \longrightarrow Hom^+_{M^s}(\overline{V}_{M^s}, \overline{V}_{M^s}) \longrightarrow K^0_{M^s} \xrightarrow{+1}. \tag{12}$$

Claim 3.4. $\mathcal{H}^0(\mathrm{id}) : \mathcal{O}_{M^s} \to \mathcal{H}^0(\mathcal{K}^0_{M^s}[-1]) \simeq \mathrm{Hom}_{SF}(\overline{\mathcal{E}}_s, \overline{\mathcal{E}}_s)$ is isomorphic.

Proof. Let s be a point in M^s. From triangles $(11)\underline{\otimes}k(s)$ and $(12)\underline{\otimes}k(s)$ one can check that $\mathcal{H}^i\left(Mc(\mathrm{id})\underline{\otimes}k(s)\right) = 0$ when $i \leq 0$ since $\overline{\mathcal{E}}_{M^s}$ is a flat family of Δ-stable and accordingly simple SFs. Because $RHom_{X_{M^s}/M^s}(\overline{E}_{M^s}, \overline{E}_{M^s})$ is perfect, also $\mathcal{K}^0_{M^s}$ is. Thus [13, Thm. 22.5] verifies $\mathcal{H}^i(Mc(\mathrm{id}))) = 0$ when $i \leq 0$. □

By this claim, this proposition is shown similarly to [7, Prop. 4.6.2.]. □

Here we mention the infinitesimal deformation of a SF $\mathcal{G} = (G, \Gamma_\bullet)$ which satisfies $H^i(G) = 0$ when $i > 0$; it is a variation of the standard deformation theory of sheaves ([15], [19] and others). Define a functor $\mathcal{D}$ from the category of Artinian local k-algebras to that of sets by

$$\mathcal{D}(A) = \left\{\mathcal{G}_A \;\middle|\; \text{an } A\text{-flat family of SFs such that } \mathcal{G}_A \otimes k \simeq \mathcal{G}\ \right\}/\simeq$$

and $\mathcal{D}(f : A \to A')(\mathcal{G}_A) = f^*\mathcal{G}_A$. If we put $V = H^0(G)$, we have the following commutative diagram whose rows and columns are triangles:

$$\begin{tikzcd}
k \arrow[r, "\mathrm{id}"] \arrow[d] & RHom_X(G,G) \arrow[r] \arrow[d, "\phi"] & RHom_X(G,G)/k \arrow[r, "+1"] \arrow[d, "\phi_+"] & {} \\
Hom^-(V,V) \arrow[r] \arrow[d] & Hom(V,V) \arrow[r] \arrow[d] & Hom^+(V,V) \arrow[r, "+1"] \arrow[d] & {} \\
Hom^-(V,V)/k \arrow[r] \arrow[d, "+1"] & Mc(\phi) \arrow[r, "\alpha"] \arrow[d, "+1"] & Mc(\phi_+) \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd} \tag{13}$$

where ϕ is the map (10). Let $A \to \overline{A}$ be a small extension of Artinian local rings, that is, a surjective ring homomorphism whose kernel $\mathfrak{a}$ satisfies $\mathfrak{a} \cdot m_A = 0$.

Lemma 3.5. *Let $\mathcal{G}_A$ and $\mathcal{G}'_A$ be elements in $\mathcal{D}(A)$ endowed with an isomorphism $\overline{\kappa} : \mathcal{G}_A \otimes \overline{A} \simeq \mathcal{G}'_A \otimes \overline{A}$. Then there is an obstruction class* $\mathrm{ob}(\overline{\kappa}, \mathfrak{a}) \in \mathcal{H}^0(Mc(\phi_+)) \otimes \mathfrak{a}$ *with the property that* $\mathrm{ob} = 0$ *if and only if $\overline{\kappa}$ extends to an isomorphism $\kappa : \mathcal{G}_A \simeq \mathcal{G}'_A$. Conversely, let $\mathcal{G}_A$ be an A-flat family of SFs extending $\mathcal{G}$. For any $v \in H^0(Mc(\phi_+)) \otimes \mathfrak{a}$ we have an A-flat family of SFs $\mathcal{G}'_A$ and an isomorphism $\overline{\kappa} : \mathcal{G}_A \otimes \overline{A} \simeq \mathcal{G}'_A \otimes \overline{A}$ such that* $\mathrm{ob}(\overline{\kappa}, \mathfrak{a}) = v$.

Proof. We shall utilize methods in [8] or [7, Section 2.A.6]. The sheaf $G_0 := G'_A \otimes k$ has an injective resolution $0 \to G_0 \xrightarrow{\epsilon_0} I^0 \xrightarrow{d_0} I^1 \to \dots$. One can find an exact sequence

$$0 \longrightarrow G'_A \xrightarrow{\epsilon'} I^0 \otimes A \xrightarrow{d'_A} I^1 \otimes A \longrightarrow \dots$$

such that $\epsilon' \otimes k = \epsilon_0$ and $d'_A \otimes k = d_0$, and an exact sequence

$$0 \longrightarrow G_A \xrightarrow{\epsilon} I^0 \otimes A \xrightarrow{d_A} I^1 \otimes A \longrightarrow \dots$$

such that $(\epsilon' \otimes \overline{A}) \circ \overline{\kappa} = \epsilon \otimes \overline{A}$ and $d_A \otimes \overline{A} = d'_A \otimes \overline{A}$. Then $\partial = d_A - d'_A : I^\bullet \to I^{\bullet+1} \otimes \mathfrak{a}$ lies in $Z^1\left(\mathrm{Hom}^\bullet_X(I^\bullet, I^\bullet) \otimes \mathfrak{a}\right)$. Since $\mathcal{H}^1\left(\mathrm{Hom}^\bullet(\Gamma(I^\bullet), \Gamma(I^\bullet)) \otimes \mathfrak{a}\right) = \mathrm{Ext}^1(V,V) \otimes \mathfrak{a}$ is zero, $\Gamma(\partial)$ belongs to $B^1(\mathrm{Hom}^\bullet(\Gamma(I^\bullet), \Gamma(I^\bullet)) \otimes \mathfrak{a})$, in other words, $\Gamma(\partial) = -\Gamma(d)e + e\Gamma(d)$ with some $e \in \mathrm{Hom}^0(\Gamma(I^\bullet), \Gamma(I^\bullet)) \otimes \mathfrak{a}$. Therefore, the diagram

$$\begin{tikzcd}
\Gamma(I^\bullet \otimes A) \arrow[r, "\Gamma(d_A)"] \arrow[d, "1-e"'] & \Gamma(I^{\bullet+1} \otimes A) \arrow[d, "1-e"] \\
\Gamma(I^\bullet \otimes A) \arrow[r, "\Gamma(d'_A)"] & \Gamma(I^{\bullet+1} \otimes A)
\end{tikzcd}$$

is commutative and induces a map $1 - e : p_{X*}(G_A) \to P_{X*}(G'_A)$. We can choose e so that this $1 - e$ commutes with flag structures because $\Gamma(\overline{\kappa})$ does. One can verify that

$$(-e, \partial) \in Z^0\left(Mc(p_{X*} : \mathrm{Hom}^\bullet_X(I^\bullet, I^\bullet) \to \mathrm{Hom}^\bullet(\Gamma(I^\bullet), \Gamma(I^\bullet)))\right) \otimes \mathfrak{a}$$

and hence obtains $[(-e, \partial)] \in \mathcal{H}^0(Mc(p_{X*})) \otimes \mathfrak{a} \simeq \mathcal{H}^0(Mc(\phi)) \otimes \mathfrak{a}$. Its image by α in (13), $\alpha[(-e, \partial)] \in \mathcal{H}^0(Mc(\phi_+)) \otimes \mathfrak{a}$, is independent of the choice of d_A, d'_A and e,

and equals zero if and only if $\overline{\kappa}$ extends to an isomorphism $\kappa : \mathcal{G}_A \simeq \mathcal{G}'_A$; its proof is left to the reader. As to the "Conversely" part, one can prove it by reversing the construction above. □

Corollary 3.6. *Let $\pi : A \to \overline{A}$ be a small extension.*

(i) *For $\mathcal{G}_{\overline{A}} \in \mathcal{D}(\overline{A})$, there is a class $\mathrm{ob}(\mathcal{G}_{\overline{A}}, \mathfrak{a}) \in \mathcal{H}^1(Mc(\phi_+)) \otimes \mathfrak{a} \simeq \mathrm{Ext}^2_X(G,G) \otimes \mathfrak{a}$ with the property that $\mathrm{ob} = 0$ if and only if some $\mathcal{G}_A \in \mathcal{D}(A)$ satisfies $\mathcal{G}_A \otimes \overline{A} \simeq \mathcal{G}_{\overline{A}}$.*

(ii) *Suppose $\mathcal{G}$ is a simple SF. Then the fiber $\mathcal{D}(\pi)^{-1}(\mathcal{G}_{\overline{A}})$ is an affine space with the transformation group $\mathcal{H}^0(Mc(\phi_+)) \otimes \mathfrak{a}$ unless it is empty.*

Proof. Since $p_{X*}(G_{\overline{A}})$ is a locally-free $\overline{A}$-module, (i) follows from deformation theory of sheaves. (ii) results from Claim 3.4 and Lemma 3.5. □

4. Set of Δ_--semistable and not Δ_+-semistable SFs

We hereafter assume that $\mathfrak{f}$ has the property (A), and parameters $\Delta_\pm$ and Δ_0 meet the conditions in Corollary 2.7. M_+ and M_- mean $M(\Delta_+, \mathfrak{f})$ and $M(\Delta_-, \mathfrak{f})$ for short. Since $M(\Delta_-, \mathfrak{f}) = M^s(\Delta_-, \mathfrak{f})$ by the remark after Definition 2.5, there is a functor $\underline{P} : (\mathrm{Sch}/M_-)^\circ \to (\mathrm{Sets})$ which associates $q : S \to M_-$ with the set of all isomorphic classes of S-flat families $\tau : \mathcal{F}^{(l)}_S \to q^*\overline{\mathcal{E}}_{M_-}$ of Δ_+-destabilizers, that is, τ is a homomorphism of flat families of SFs such that, for any point $s \in S$, $\tau \otimes k(s) : \mathcal{F}^{(l)}_s \to \overline{\mathcal{E}}_s$ gives a Δ_+-destabilizer of $\overline{\mathcal{E}}_s$.

Lemma 4.1. *A closed subscheme $P \subset M_-$ represents the functor $\underline{P}$.*

Proof. $\underline{P}$ is representable by Grothendieck's Quot-schemes and [11, Lem. 3.1]. If $\mathcal{F}^{(l)}$ is a Δ_+-destabilizer of a Δ_--semistable SF $\mathcal{E}$ of type $\mathfrak{f}$, then $\mathrm{Hom}_{SF}(\mathcal{F}^{(l)}, \mathcal{E}/\mathcal{F}^{(l)}) = 0$ by Proposition 2.6. Hence the same argument as in the proof of [20, Lem. 2.2] shows this lemma. □

There are P-flat families of SFs $\mathcal{F}^{(l)}_P = (F^{(l)}_P, \Gamma^{(l)}_{\bullet,P})$ and $\mathcal{F}^{(r)}_P = (F^{(r)}_P, \Gamma^{(r)}_{\bullet,P})$, and an exact sequence of families of SFs

$$0 \longrightarrow \mathcal{F}^{(l)}_P \longrightarrow \overline{\mathcal{E}}_{M_-}|_P \longrightarrow \mathcal{F}^{(r)}_P \longrightarrow 0. \tag{14}$$

Now let $T \subset \overline{T}$ be a closed immersion whose ideal sheaf $\mathfrak{a} \subset \mathcal{O}_{M_-}$ satisfies that $\mathfrak{a}^2 = 0$, and $f : \overline{T} \to M_-$ a morphism such that its restriction to T factors through $P \subset M_-$, in other words, $f|_T$ induces a morphism $g : T \to P$. When we denote $f^*(\overline{E}_{M_-}) = \overline{E}_{\overline{T}}$, $g^* F^{(l)}_P = F^{(l)}_T$ and so on, the exact sequence of $\mathcal{O}_{X_{\overline{T}}}$-modules

associated with (14) gives a diagram in $\mathrm{Coh}(X_{\overline{T}})$

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathfrak{a} \otimes_T F_T^{(l)} \arrow[r] & \mathfrak{a} \otimes_T \overline{E}_{\overline{T}} \arrow[r] \arrow[d] & \mathfrak{a} \otimes_T F_T^{(r)} \arrow[r] & 0 \\ & & \overline{E}_{\overline{T}} \arrow[d] & & \\ 0 \arrow[r] & F_T^{(l)} \arrow[r] & \overline{E}_{\overline{T}}|_T \arrow[r] \arrow[d] & F_T^{(r)} \arrow[r] & 0 \\ & & 0 & & \end{tikzcd} \tag{15}$$

whose rows and columns are exact. This diagram induces the following:
(i) An $\mathcal{O}_{X_T}$-module $W_T = \mathrm{Ker}(\overline{E}_{\overline{T}} \to F_T^{(r)}) / \mathrm{Im}(\mathfrak{a} \otimes F_T^{(l)} \to \overline{E}_{\overline{T}})$ and an exact sequence

$$0 \longrightarrow \mathfrak{a} \otimes F_T^{(r)} \longrightarrow W_T \longrightarrow F_T^{(l)} \longrightarrow 0. \tag{16}$$

Similarly, homomorphisms of $\mathcal{O}_T$-modules with flag structures associated with (14) brings following elements:
(ii) An $\mathcal{O}_T$-module $\Lambda_{\bullet,T} = \mathrm{Ker}(\overline{\Gamma}_{\bullet,\overline{T}} \to \Gamma_{\bullet,T}^{(r)}) / \mathrm{Im}(\mathfrak{a} \otimes \Gamma_{\bullet,T}^{(l)} \to \overline{\Gamma}_{\bullet,\overline{T}})$ and an exact sequence

$$0 \longrightarrow \mathfrak{a} \otimes \Gamma_{\bullet,T}^{(r)} \longrightarrow \Lambda_{\bullet,T} \longrightarrow \Gamma_{\bullet,T}^{(l)} \longrightarrow 0\,; \tag{17}$$

(iii)Homomorphisms $\tau_{\bullet-1} : \Lambda_{\bullet-1,T} \to \Lambda_{\bullet,T}$ and $\iota_\bullet : \Lambda_{\bullet,T} \to p_{X*}(W_T)$ such that the diagram

$$\begin{tikzcd} \mathfrak{a} \otimes \Gamma_{\bullet-1,T}^{(r)} \arrow[r, hook] \arrow[d] & \mathfrak{a} \otimes \Gamma_{\bullet,T}^{(r)} \arrow[r, hook] \arrow[d] & \mathfrak{a} \otimes p_{X*}(F_T^{(r)}) \arrow[d] \\ \Lambda_{\bullet-1,T} \arrow[r, "\tau_{\bullet-1}"] \arrow[d] & \Lambda_{\bullet,T} \arrow[r, "\iota_\bullet"] \arrow[d] & p_{X*}(W_T) \arrow[d] \\ \Gamma_{\bullet-1,T}^{(l)} \arrow[r, hook] & \Gamma_{\bullet,T}^{(l)} \arrow[r, hook] & p_{X*}(F_T^{(l)}), \end{tikzcd}$$

which is a combination of $p_{X*}((16))$, (17) and flag structures of $\mathcal{F}_P^{(l)}$ and $\mathcal{F}_P^{(r)}$, is commutative.

Lemma 4.2. *$f : \overline{T} \to M_-$ factors through $P \subset M_-$ if and only if there are a section $\kappa : F_T^{(l)} \to W_T$ of (16) and a section $\kappa_\bullet : \Gamma_{\bullet,T}^{(l)} \to \Lambda_{\bullet,T}$ of (17) which make*

the following diagram commutative.

$$\begin{tikzcd} \Lambda_{\bullet-1,T} \arrow[r, "\tau_{\bullet-1}"] & \Lambda_{\bullet,T} \arrow[r, "\iota_\bullet"] & p_{X*}(W_T) \\ \Gamma^{(l)}_{\bullet-1,T} \arrow[u, "\kappa_{\bullet-1}"] \arrow[r, hook] & \Gamma^{(l)}_{\bullet,T} \arrow[u, "\kappa_\bullet"] & p_{X*}(F^{(l)}_T) \arrow[u, "p_{X*}(\kappa)"] \arrow[l, hook'] \end{tikzcd}$$

Proof. It is a variation of the deformation theory of Quot-schemes [7, page 43], so we omit the proof. □

We shall denote $V^{(l)}_P = p_{X*}(F^{(l)}_P)$, $V^{(r)} = p_{X*}(F^{(r)}_P)$ and $\omega_X = K_X[2]$. By the duality theorem [5], a natural morphism

$$\psi_- : RHom_{X_P/P}(F^{(l)}_P, F^{(r)}_P) \longrightarrow Hom^+_P(V^{(l)}_P, V^{(r)}_P) \tag{18}$$

induces a triangle in $D^b(P)$

$$Hom^+_P(V^{(l)}_P, V^{(r)}_P)^\vee \longrightarrow Hom_{X_P/P}(F^{(r)}_P, F^{(l)}_P(\omega_X)) \longrightarrow \mathcal{K} \xrightarrow{+1}. \tag{19}$$

Lemma 4.3. *There is an obstruction class* $\mathrm{ob}(f,g) \in \mathrm{Ext}^1_T(Lg^*(\mathcal{K}), \mathfrak{a})$ *with the property that* $f : \overline{T} \to M_-$ *factors through* $P \subset M_-$ *if and only if* $\mathrm{ob} = 0$.

Proof. The functor $R\,\mathrm{Hom}_T(Lg^*(?), \mathfrak{a})$ takes (19) to a triangle

$$R\,\mathrm{Hom}_T(Lg^*\mathcal{K}, \mathfrak{a}) \longrightarrow R\,\mathrm{Hom}_{X_T}(F^{(l)}_T, F^{(r)}_T \underset{T}{\otimes} \mathfrak{a}) \xrightarrow{\psi_-} R\Gamma_T(Hom^+_T(V^{(l)}_T, V^{(r)}_T \otimes \mathfrak{a})) \xrightarrow{+1}. \tag{20}$$

Let $\epsilon^{(r)} : F^{(r)}_T \otimes \mathfrak{a} \to (I^{\bullet(r)}, d^{(r)})$ be an injective resolution in $\mathrm{Mod}(X_T)$. As for a $\mathcal{O}_T$-module $V^{(r)}_T \otimes \mathfrak{a}$ with the filtration $\Gamma^{(r)}_{1,T} \otimes \mathfrak{a} \subset \dots \Gamma^{(r)}_{n,T} \otimes \mathfrak{a} \subset \Gamma^{(r)}_{n+1,T} \otimes \mathfrak{a} = V^{(r)}_T \otimes \mathfrak{a}$, pick an injective resolution $\mathrm{gr}^i(V^{(r)}_T \otimes \mathfrak{a}) \to (K^\bullet_i, d_i)$ for $i = 1, \dots, n+1$ and find an injective resolution $V^{(r)}_T \otimes \mathfrak{a} \to (K^\bullet = \oplus_{j=1}^{n+1} K^\bullet_j, d_K)$ such that $d_K(\oplus_{j\le i} K^\bullet_j) \subset \oplus_{j\le i} K^\bullet_j$ and that $\mathrm{gr}^i(d_K) : K^\bullet_i \to K^{\bullet+1}_i$ coincides with d_i for every i. In particular $(K^\bullet, d_K)$ is a filtered complex. One can describe $R\,\mathrm{Hom}_T(Lg^*\mathcal{K}, \mathfrak{a})$ by $I^{\bullet(r)}$ and $K^\bullet$. Indeed, a natural map $V^{(r)}_T \otimes \mathfrak{a} \to p_{X*}(F^{(r)}_T \otimes \mathfrak{a})$ is isomorphic, and its inverse map extends to a quasi-isomorphism

$$\nu : (p_{X*}(I^\bullet), p_{X*}(d_I)) \longrightarrow (K^\bullet, d_K). \tag{21}$$

Fix an affine open covering $\{T_a\}$ of T such that the exact sequence $p_{X*}((16))|_{T_a}$,

$$0 \longrightarrow \mathfrak{a} \otimes V^{(r)}_T|_{T_a} \longrightarrow p_{X*}(W_T)|_{T_a} \longrightarrow V^{(l)}_T|_{T_a} \longrightarrow 0,$$

has a section $j_a : V^{(l)}_T|_{T_a} \to p_{X*F}(W_T)|_{T_a}$ which preserves filtrations $\Gamma^{(l)}_{\bullet T}$ and $\Lambda_{\bullet T}$. Since $K^\bullet$ has a filtration, we obtain complexes $Hom^+_T(V^{(l)}_T, K^\bullet)$ and

$$(C^\bullet(\{T_a\}, Hom^+_T(V^{(l)}_T, K^\bullet)), (-1)^{\deg} d_{Cech} + d_K),$$

where $(C^\bullet(\{T_a\}, Hom^+_T(V^{(l)}_T, K^q)), d_{Cech})$ is the Cěch complex. The homomorphism ν (21) derives

$$p_{X*}(\nu) : \mathrm{Hom}_{X_T}(F^{(l)}_T, I^{\bullet(r)}) \longrightarrow C^\bullet(\{T_a\}, Hom^+_T(V^{(l)}_T, K^\bullet)),$$

and $R\,\mathrm{Hom}_T(Lg^*\mathcal{K}, \mathfrak{a})[1]$ at (20) is represented by $Mc(p_{X*}(\nu))$.

Let $\alpha \in Z^1(\mathrm{Hom}_{X_T}(F_T^{(l)}, I^\bullet))$ represent the image of identity map by the map $\mathrm{Hom}_{X_T}(F_T^{(l)}, F_T^{(l)}) \to \mathrm{Ext}^1_{X_T}(F_T^{(l)}, F_T^{(r)} \otimes \mathfrak{a})$ coming from (16). By the exact sequence (16), $\epsilon^{(r)} : \mathfrak{a} \otimes F_T^{(r)} \hookrightarrow I^{0\,(r)}$ extends to $\epsilon' : W_T \to I^{0\,(r)}$. If we denote $V_T^{(l)}|_{Ta} \xrightarrow{j_a} p_{X*}(W_T)|_{T_a} \xrightarrow{\epsilon'} p_{X*}(I^{0(r)}) \xrightarrow{\nu} K^0$ by i_a, then one can check that $(\alpha, \{\overline{i_a}\}) \in [Mc(p_{X*}(\nu))]_0$ is contained in $Z^0(Mc(p_{X*F}(\nu)))$ and that

$$\mathrm{ob}(f,g) := [(\alpha, \{\overline{i_a}\})] \in H^0(Mc(p_{X*}(\nu))) \simeq \mathrm{Ext}^1_T(Lg^*\mathcal{K}, \mathfrak{a})$$

enjoys the property asserted in this lemma. $\square$

When sheaves G and G' on a scheme S have filtrations $\{G_i \subset G\}$ and $\{G'_i \subset G'\}$ of length n, we have objects $R\,\mathrm{Hom}^-_S(G, G')$ and $R\,\mathrm{Hom}^+_X(G, G')$ in $D(S)$ with a triangle

$$R\,\mathrm{Hom}^-_S(G, G') \longrightarrow R\,\mathrm{Hom}_S(G, G') \longrightarrow R\,\mathrm{Hom}^+_S(G, G') \xrightarrow{+1}; \tag{22}$$

see [7, p. 49]. For a point $s \in P$ corresponding to a SF $\mathcal{E}$, we here explain how to derive the following diagram whose rows and columns are triangles:

$$\begin{tikzcd}
R\,\mathrm{Hom}^{(-)}_X(E,E)/k \arrow[r] \arrow[d, "\psi_-"] & R\,\mathrm{Hom}_X(E,E)/k \arrow[r] \arrow[d, "\phi_+"] & R\,\mathrm{Hom}_X(F^{(l)}, F^{(r)}) \arrow[r, "+1"] \arrow[d, "\psi_+"] & {} \\
\mathrm{Hom}^{+(-)}(V,V) \arrow[r] \arrow[d] & \mathrm{Hom}^+(V,V) \arrow[r] \arrow[d] & \mathrm{Hom}^+(V^{(l)}, V^{(r)}) \arrow[r, "+1"] \arrow[d] & {} \\
Mc(\psi_-) \arrow[r] \arrow[d, "+1"] & Mc(\phi_+) \arrow[r] \arrow[d, "+1"] & Mc(\psi_+) \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd} \tag{23}$$

Equation (14) gives filtrations $F^{(l)} \subset E$ and $\Gamma(F^{(l)}) \subset V = \Gamma(E)$, and the flag structures of SFs are nothing but filtrations $\Gamma_\bullet \subset V$, $\Gamma^{(l)}_\bullet \subset V^{(l)} = \Gamma(F^{(l)})$, and so on. $R\,\mathrm{Hom}^{(-)}_X(E,E)$ means $R\,\mathrm{Hom}^-_X$ with respect to the former filtration, and the first row in (23) comes from (22). $\mathrm{Hom}^+(V,V)$ means, by the definition, Hom^+ with respect to the latter filtration, and $\mathrm{Hom}^{+(-)}$ the kernel of a natural map $\mathrm{Hom}^+(V,V) \to \mathrm{Hom}^+(V^{(l)}, V^{(r)})$. Morphisms ϕ_+ and ψ_- are those of (13) and (18), and ψ_+ the induced one.

Proposition 4.4. *The tangent space T_sP is isomorphic to $H^0(Mc(\psi_+))$.*

Proof. Since $H^{-1}(Mc(\psi_+)) \simeq \mathrm{Hom}_{SF}(\mathcal{F}^{(l)}, \mathcal{F}^{(r)}) = 0$, (23) induces an exact sequence

$$0 \longrightarrow H^0(Mc(\psi_-)) \longrightarrow H^0(Mc(\phi_+)) \xrightarrow{\varphi} H^0(Mc(\psi_+)).$$

If $f : \overline{T} = \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \to M_-$ and $f' : \overline{T} \to P \subset M_-$ extend $g = s : T = \mathrm{Spec}\, k \to M_-$, then φ sends $\mathrm{ob}(\overline{\kappa}, k\cdot\epsilon) \in H^0(Mc(\psi_+)) \otimes k \cdot \epsilon$ associated with $\overline{\kappa} : f^*\mathcal{E}_{M_-} \otimes k(s) = \mathcal{E} = f'^*\mathcal{E}_{M_-} \otimes k(s)$ to $\mathrm{ob}(f,g) \in \mathrm{Ext}^1(Lg^*(\mathcal{K}), k\cdot\epsilon) \simeq H^0(Mc(\psi_+)) \otimes k\cdot\epsilon$, where the last equality holds from (20). It immediately leads to this proposition. $\square$

Corollary 4.5. *Let r be an integer and C a compact subset in the ample cone of X. If $\mathcal{O}(1)$ lies in C and if $s \in P$ corresponds to a SF $\mathcal{E} = (E, \Gamma_\bullet)$ with $\mathrm{rk}(E) = r$, then it holds that $\mathrm{codim}_s(P, M_-) \geq \Delta(E)/2r - B(r, X, C)$, where $\Delta(E) = 2rc_2(E) - (r-1)c_1(E)^2$ and $B(r, X, C)$ is a constant depending only on (r, X, C).*

Proof. By the proposition above and Corollary 3.6,

$$\dim_s P \leq \dim H^0(Mc(\psi_+)) \leq \dim \mathrm{Ext}_X^{1(-)}(E,E) - 1 + \dim \mathrm{Hom}^+(V,V) \quad \text{and}$$
$$\dim_s M_- \geq \dim H^0(Mc(\phi_+)) - \dim H^1(Mc(\phi_+)) = \dim \mathrm{Hom}^+(V,V) - \chi(E,E) + 1.$$

Then this corollary results from O'Grady's estimation of $\dim \mathrm{Ext}^{1(-)}$ ([7, Prop 3.A.2] and [16]) and the Riemann-Roch formula. □

5. Blowing-up construction

As Corollary 4.5 shows, it is reasonable to expect that M_- and M_+ are birationally equivalent. We here describe how to connect them by a single blowing-up and down in a moduli-theoretic way. Let $p : \tilde{M} \to M_-$ be the blowing-up along P with exceptional divisor E. Then we have a flat family of SFs $p^*\overline{\mathcal{E}}_{M_-} = \overline{\mathcal{E}}_{\tilde{M}}$ over $\tilde{M}$ and an exact sequence of flat families of SFs

$$0 \longrightarrow p^*\mathcal{F}_P^{(l)} = \mathcal{F}_E^{(l)} \longrightarrow \overline{\mathcal{E}}_{M_-}|_E \longrightarrow \mathcal{F}_E^{(r)} \longrightarrow 0$$

coming from (14), and we can show the following facts in the same way as the case of rank-two sheaves ([20, Section 3 and 4]) except for obvious modifications:
(i) $\mathcal{E}'_{\tilde{M}} := \mathrm{Ker}(\overline{\mathcal{E}}_{\tilde{M}} \to \overline{\mathcal{E}}_{\tilde{M}}|_E \to \mathcal{F}_E^{(r)})$ is a flat family of SFs over $\tilde{M}$ equipped with an exact sequence

$$0 \longrightarrow \mathcal{F}_E^{(r)} \otimes \mathcal{O}_E(-E) \xrightarrow{k_1} \mathcal{E}'_{\tilde{M}}|_E \longrightarrow \mathcal{F}_E^{(l)} \longrightarrow 0 \tag{24}$$

of families of SFs over E. One can regard this as an elementary transform of families of SFs.
(ii) When one applies results in the last section to case where $f : \mathrm{Spec}(\mathcal{O}_{\tilde{M}}/\mathcal{O}(-2E)) = \overline{T} \xrightarrow{p} M_-$ and $g : E = T \xrightarrow{p} P$, he obtains $\mathcal{W}_E = (W_E, \Lambda_{\bullet E})$, which is a flat family of SFs since $\mathfrak{a} = \mathcal{O}_E(-E)$ is a line bundle, and an exact sequence

$$0 \longrightarrow \mathcal{F}_E^{(r)} \otimes \mathcal{O}_E(-E) \xrightarrow{k_2} \mathcal{W}_E \longrightarrow \mathcal{F}_E^{(l)} \longrightarrow 0. \tag{25}$$

In fact, there is an isomorphism $\lambda : \mathcal{E}'_{\tilde{M}}|_E \simeq \mathcal{W}_E$ of families of SFs which satisfies $\lambda \circ k_1 = k_2$ in (24) and (25).
(iii) For any point $s \in P$, the exact sequence (24) $\otimes\, k(s)$ of SFs is nontrivial. Consequently $\mathcal{E}'_{M_-}$ is a family of Δ_+-semistable SFs by Corollary 2.7 and so results in a morphism $q : \tilde{M} \to M_+$.

Proposition 5.1. *By reversing Δ_- and Δ_+ we get a closed subscheme $P' \subset M_+$ provided with a similar property to $P \subset M_-$, and then the morphism $q : \tilde{M} \to M_+$ defined above is the blowing-up of M_+ along P'. Consequently*

$$M_- \xleftarrow{p} \tilde{M} \xrightarrow{q} M_+$$

are blowing-ups derived from moduli theory.

We shall end this article with relating variation of parameters Δ and the Δ-stability of SFs to that of polarizations H on X and the H-semistability of sheaves. When a class $\mathfrak{c} = (r, c_1, c_2) \in \mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}$ is given, let H_- and H_+ be polarizations on X contained in adjacent chambers of type $\mathfrak{c}$, and $H_0 = tH_- + (1-t)H_+$ $(0 < t < 1)$ lie in just one wall of type $\mathfrak{c}$; see [21, Def. 2.1] for chambers and walls of type $\mathfrak{c}$. For positive integers m, n and a constant $0 < a < 1$,

$$\chi^a(E)(l) = (1-a)\chi(E \otimes mH_-(l)) + a\chi(E \otimes nH_+(l))$$

defines a-*(semi)stability* of a sheaf E on X.

Proposition 5.2. *If $m \gg 0$ and if $n \gg 0$ with respect to m, then the following holds about a sheaf E of type $\mathfrak{c}$: E is H_--stable if and only if E is 0-stable if and only if E is a_--stable where $a_- > 0$ is sufficiently small. This also holds when one replaces "stable" with "semistable" here.*

Proof. As to the first "if and only if" part, refer to [1, Lem. 3.1] in rank-two case and [12, Lem. 3.6] for general case. The second is an easy exercise. □

Set $\mathcal{O}(1)$, L and C to be H_0, nH_+ and $nH_+ - mH_-$ respectively. We can assume that $\mathfrak{f} = (\chi(E(l)), \chi(E \otimes L(-C)(l)), \chi(E \otimes L(l)), l_1, \ldots, l_n)$, where E is of type $\mathfrak{c}$, has the property (A). Choose a parameter Δ_{H_-} (resp. Δ_{H_+}) so that no SF-wall of type $\mathfrak{f}$ separates Δ_{H_-} and $(0, 0, \ldots, 0)$ (resp. Δ_{H_+} and $(1, 0, \ldots, 0)$). Then a sheaf E of type $\mathfrak{c}$ and a SF $\mathcal{E} = (E, \Gamma_\bullet)$ of type $\mathfrak{f}$ satisfies that (i) E is H_--semistable if $\mathcal{E}$ is Δ_{H_-}-semistable and that (ii) $\mathcal{E}$ is Δ_{H_-}-stable if E is H_--stable. Thus, if one denotes by $M(\Delta_{H_-}, \mathfrak{c}) \subset M(\Delta_{H_-}, \mathfrak{f})$ the union of connected components consisting of all SFs $\mathcal{E} = (E, \Gamma_\bullet)$ such that E is of type $\mathfrak{c}$, then one gets a natural morphism to the coarse moduli scheme $M(H_-, \mathfrak{c})$ of H_--semistable sheaves of type $\mathfrak{c}$, $h : M(\Delta_{H_-}, \mathfrak{c}) \to M(H_-, \mathfrak{c})$, whose restriction $h : h^{-1}(M^s(H_-, \mathfrak{c})) \to M^s(H_-, \mathfrak{c})$ is a Grassmannian-bundle in étale topology. Because there is a sequence of parameters $\Delta_{H_-} = \Delta_0, \Delta_1, \ldots, \Delta_m = \Delta_{H_+}$ such that Δ_i and Δ_{i+1} are in adjacent chambers of type $\mathfrak{f}$ for all i, we arrive at a diagram

$$\begin{tikzcd}
 & \tilde{M}_0 \arrow[dl, "p_0"'] \arrow[dr, "q_0"] & & \cdots \arrow[dl, "p_1"'] & \tilde{M}_{m-1} \arrow[dr, "q_{m-1}"] & \\
M(\Delta_0, \mathfrak{c}) \arrow[d, "h"] & & M(\Delta_1, \mathfrak{c}) & & & M(\Delta_m, \mathfrak{c}) \arrow[d, "h"] \\
M(H_-, \mathfrak{c}) & & & & & M(H_+, \mathfrak{c}),
\end{tikzcd}$$

where p_i and q_i are blowing-ups in Proposition 5.1.

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Department of Mathematics, Faculty of science and technology, Sophia Universiy, 7-1, Kioi-cho, Chiyoda-ku, Tokyo, Japan

E-mail address: yamada@mm.sophia.ac.jp